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Graded solutions of the Yang–Baxter relation and link polynomials

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Abstract. From a family of graded solvable models we derive representations of the braid group associated with the Lie superalgebra $\mathrm{gl}(M|N)$ and give explicitly a general form of the Markov traces on the representations. The braid operators thus obtained satisfy the Hecke algebra. We construct composite solvable models and obtain link polynomials from the braid operators for the composite models.

1. Introduction

It is well known that the Yang–Baxter relation is a sufficient condition for the solvability of models in statistical mechanics and field theories [1–7]. The Boltzmann weight $w(a,b,c,d;u)$ of a vertex model is defined on the configuration $\{a,b,c,d\}$ of edge variables round a vertex (figure 1). For vertex models the Yang–Baxter relation is

$$\sum_{abc} w(b,c,q,r;u)w(a,k,p,c;u+v)w(i,j,a,b;v) = \sum_{abc} w(a,b,p,q;v)w(i,c,a,r;u+v)w(j,k,b,c;u). \tag{1}$$

Here, the parameter u is called the spectral parameter, which represents the strength and anisotropy of the coupling. We define the Yang–Baxter operator (construction unit of the diagonal-to-diagonal transfer matrix) for vertex models by

$$X_i(u) = \sum_{abcd} X^{ab}_{cd}(u) I^{(1)} \otimes \cdots \otimes e^{(i)}_{ac} \otimes e^{(i+1)}_{bd} \otimes I^{(i+2)} \otimes \cdots \otimes I^{(n)}. \tag{2}$$

Here $X^{ab}_{cd}(u) = w(c,d,b,a;u)$, $I^{(i)}$ denotes the identity matrix and e_{ab} a matrix such that $(e_{ab})_{jk} = \delta_{ja}\delta_{kb}$. We note that the operators $\{X_i(u)\}$ act on the tensor product

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space $V^{(1)} \otimes V^{(2)} \otimes \cdots \otimes V^{(n)}$. In terms of the Yang–Baxter operators $X_i(u)$ the Yang–Baxter relation (1) is written as (Yang–Baxter algebra) [1, 3, 8, 9]

$$\begin{aligned} &X_i(u)X_{i+1}(u+v)X_i(v) = X_{i+1}(v)X_i(u+v)X_{i+1}(u) \\ &X_i(u)X_j(v) = X_j(v)X_i(u) \qquad |i-j| \geq 2. \end{aligned} \tag{3}$$

The Yang–Baxter relation in this form has an advantage in that we can easily see the connection of solvable models to the braid group.

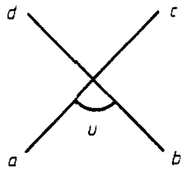

Figure 1. Boltzmann weight of vertex model w(a, b, c, d; u).

Recently, the Yang–Baxter relation has been found to be a key to several fields in mathematical physics. First, the relation is an extension of the defining relations of the braid group; we can construct link polynomials, topological invariants for knots and links, from solutions of the Yang–Baxter relation. A general method to construct braid group representations and link polynomials from exactly solvable models has been established [8–23]. We can formulate various link polynomials [24–28] by this method and obtain new link polynomials [8, 9, 11, 12, 19, 22]. From this viewpoint the relation is a tool for knots and links [18]. Second, the representations of the braid group can be considered as extensions of representations of the symmetric group. They are related to interesting mathematical objects, such as the Temperley–Lieb algebra, Hecke algebra, C^* algebra, etc [25, 29]. Finally, braid matrices derived from exactly solvable models are closely connected to new physics such as the monodromy matrices of the Knizhnik–Zamolodchikov equation [30–32] and strange statistics [33, 34].

In this paper we construct representations of the braid group and link polynomials from a family of graded vertex models related to the Lie superalgebra gl($M|N$). The symmetry of the models is different from that for the models associated with the Lie algebra sl(M). We discuss the composite models and link polynomials derived from them.

The outline of this paper is given in the following. In section 2 we introduce vertex models associated with gl($M|N$) and explain the graded Yang–Baxter relation and the connection to the Lie superalgebra gl($M|N$). We construct composite vertex models from these models. In section 3 we derive representations of the braid group from the vertex models and obtain link polynomials by constructing the Markov traces on the representation. We discuss link polynomials obtained from composite models. In section 4 we give concluding remarks.

2. Graded vertex models

2.1. Vertex models associated with gl(M|N)

Let us introduce a family of solvable vertex models associated with gl($M|N$) [23, 35]. The models are given in the case IB in [35]. We introduce a set of signs $\{\epsilon_i\}$

$$\epsilon_i = 1 \text{ or } -1 \qquad \text{for } \ i = 1, \cdots, M+N. \tag{4}$$

Note that the sign ϵ_i represents the 'parity' of the edge state i. We also introduce the 'grade' $p(i) \in \{0,1\}$ of the edge state i as $\epsilon_i = (-1)^{p(i)}$. The number of positive (respectively negative) signs is given by M (respectively N). In this way we have introduced the graded symmetry. For any set of signs $\{\epsilon_i\}$ we have a solution of the Yang–Baxter relation. The Boltzmann weights are given as follows:

$$\begin{aligned} w(a,a,a,a;u) &= \sinh(\eta - \epsilon_a u)/\sinh\eta \\ w(a,b,b,a;u) &= \begin{cases} \exp(-u) & \text{for } a < b \\ \exp(u) & \text{for } a > b \end{cases} \\ w(a,b,a,b;u) &= \sinh u/\sinh\eta \qquad \text{for } a \neq b. \end{aligned} \tag{5}$$

Here η is a parameter and the edge variables a and b take values $1, 2, \cdots, M+N$. The models have the charge conservation property: $w(a,b,c,d;u) = 0$, unless $\boldsymbol{a}+\boldsymbol{b} = \boldsymbol{c}+\boldsymbol{d}$. Here $\boldsymbol{a}$ represents 'charge' of the state a, which is vector valued in general. The Boltzmann weights satisfy the reflection symmetry $w(a,b,c,d;u) = w(c,d,b,a;u)$. They also satisfy the standard initial condition and the inversion relation.

(i) Standard initial condition:

$$X^{ab}_{cd}(u=0) = \delta_{ac}\delta_{bd}. \tag{6}$$

(ii) Inversion relation (unitarity condition):

$$\sum_{mp} X^{ab}_{mp}(u) X^{mp}_{cd}(-u) = \rho(u)\rho(-u)\delta_{ac}\delta_{bd} \tag{7}$$

where $\rho(u) = \sinh(\eta - u)/\sinh\eta$.

It is instructive to consider simple examples of the models. For $M+N=2$, the case $(\epsilon_1, \epsilon_2) = (1,1)$ (and $(-1,-1)$) is equivalent to the 6-vertex model. The model for the case $(\epsilon_1,\epsilon_2) = (1,-1)$ (and $(-1,1)$) corresponds to the free fermion 6-vertex model [36]. We note that by changing the sign of the spectral parameter as $u \to -u$ and changing signs in the Boltzmann weights as $w(a,b,b,a;u) \to -w(a,b,b,a;u)$ $(a \neq b)$, the models for $(-1,-1)$ and $(-1,1)$ are transformed into those for $(1,1)$ and $(1,-1)$, respectively. For $M+N > 2$, the two cases: $\epsilon_i = 1$ (for all i) and $\epsilon_i = -1$ (for all i) are the M-state vertex models associated with $\mathrm{sl}(M)$ [35, 37].

Under the charge conservation property the Boltzmann weights of the models satisfy the Yang–Baxter relation after the following transformations (symmetry breaking transformations, or gauge transformations) [22, 38]:

$$\begin{aligned} X^{ab}_{cd}(u) &\to \tilde{X}^{ab}_{cd}(u) = \alpha^{ab}_{cd}(u)\beta^{ab}_{cd}\gamma^{ab}_{cd}\delta^{ab}_{cd}X^{ab}_{cd}(u) \\ \alpha^{ab}_{cd}(u) &= \exp[\boldsymbol{\mu}\cdot(\boldsymbol{b}+\boldsymbol{d}-\boldsymbol{a}-\boldsymbol{c})u] \\ \beta^{ab}_{cd} &= \exp[\boldsymbol{\nu}\cdot(\boldsymbol{a}+\boldsymbol{d}-\boldsymbol{b}-\boldsymbol{c})] \\ \gamma^{ab}_{cd} &= \exp[\zeta(\boldsymbol{a}\cdot\boldsymbol{b}-\boldsymbol{c}\cdot\boldsymbol{d})] \\ \delta^{ab}_{cd} &= \exp[\pi\sqrt{-1}(\boldsymbol{a}+\boldsymbol{c})\cdot\boldsymbol{\epsilon}] \end{aligned} \tag{8}$$

where ζ is a free parameter, $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ are vectors with arbitrary directions and magnitudes, and $\boldsymbol{\epsilon}$ is a vector such that $(\boldsymbol{a}+\boldsymbol{c})\cdot\boldsymbol{\epsilon}$ is an integer for any edge charges $\boldsymbol{a}$ and $\boldsymbol{c}$.

2.2. Reduction to rational solutions associated with gl(M|N)

We briefly introduce the notion of Lie superalgebras [39]. A Lie superalgebra is a $\mathbf{Z}_2$-graded algebra over $\mathbf{C}$. We define the grade $p(A)$ of an element A by

$$p(A) = \begin{cases} 0 & \text{if } A \text{ is bosonic} \\ 1 & \text{if } A \text{ is fermionic.} \end{cases} \tag{9}$$

The bracket given by

$$[A, B] = AB - (-1)^{p(A)p(B)} BA \tag{10}$$

satisfies the super-Jacobi identity:

$$(-1)^{p(A)p(C)}[A,[B,C]] + (-1)^{p(B)p(A)}[B,[C,A]] + (-1)^{p(C)p(B)}[C,[A,B]] = 0. \tag{11}$$

We define the tensor product $A \otimes B$ with the induced grading:

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = (-1)^{p(b_1)p(a_2)} a_1 a_2 \otimes b_1 b_2 \qquad a_i \in A, b_i \in B. \tag{12}$$

We introduce the graded permutation operator π_i as

$$\pi_i(u) = \sum_{abcd} \pi^{ab}_{cd}(u) I^{(1)} \otimes \cdots \otimes e^{(i)}_{ac} \otimes e^{(i+1)}_{bd} \otimes I^{(i+2)} \otimes \cdots \otimes I^{(n)} \tag{13}$$

where

$$\pi^{ab}_{cd} = (-1)^{p(a)p(b)} \delta_{ad} \delta_{bc}. \tag{14}$$

Let us consider the Lie superalgebra gl$(M|N)$. The generators $\{E^a_b\}$ of gl$(M|N)$ satisfy the defining relations

$$[E^a_b, E^c_d] = \delta_{cb} E^a_d - (-1)^{(p(a)+p(b))(p(c)+p(d))} \delta_{ad} E^c_b. \tag{15}$$

The parity of the generator E^a_b is given by $(-1)^{p(a)+p(b)}$. The Casimir operator C for gl$(M|N)$ is given by

$$C = \sum_{ab} (-1)^{p(b)} E^a_b E^b_a. \tag{16}$$

We return to the vertex model given in (5). The R matrix $R(u)$ in the context of the quantum inverse scattering method is related to the Yang–Baxter operator $X(u)$ by

$$R^{ab}_{cd}(u) = X^{ab}_{dc}(u) \pi^{dc}_{cd}. \tag{17}$$

The matrix elements $R^{ab}_{cd}(u)$ satisfy the graded Yang–Baxter relation [40]

$$\begin{aligned} &R^{rq}_{cb}(u) R^{cp}_{ka}(u+v) R^{ba}_{ji}(v) (-1)^{p(b)p(c)+p(a)p(k)+p(i)p(j)} \\ &\quad = R^{qp}_{ba}(v) R^{ra}_{ci}(u+v) R^{cb}_{kj}(u) (-1)^{p(a)p(b)+p(i)p(c)+p(j)p(k)}. \end{aligned} \tag{18}$$

By rescaling the variables as $u \to \epsilon u$, $\eta \to \epsilon\eta$ and taking the limit $\epsilon \to 0$, we derive a rational solution [41] of the Yang–Baxter relation from the vertex model (5). The Yang–Baxter operator $\tilde{X}_i(u)$ for the rational solution has the form

$$\tilde{X}_i(u) = I - \frac{u}{\eta} \pi_i. \tag{19}$$

Here π_i is the graded permutation operator (13). The rational solution for the R matrix satisfies the graded Yang–Baxter relation (18) [40].

Thus the vertex model (5) can be considered as an extension (q analogue) of the rational solution (19). In section 3 we shall see that the braid operator obtained from the vertex model is a q analogue of the graded permutation operator, and that the infinitesimal pure braid operator for the braid operator is equivalent to the Casimir operator (16) of gl$(M|N)$.

2.3. Fusion of the vertex model associated with gl(M|N)

By the method of fusion we can construct composite solvable models [12, 14, 42]. These models are special cases of Z-invariantly generalised inhomogeneous models [43]. In the following discussion we use only one property, that the Yang–Baxter operator $X_i(u)$ for the vertex model (5) has two eigenvalues; more precisely, has quadratic minimal polynomial. From this property the Yang–Baxter operator becomes the projection operator when the spectral parameter $u = \pm\eta$ [14]. We introduce an operator A_i by

$$A_i = X_i(u = \eta). \tag{20}$$

Then we see that the operator satisfies the relations

$$\begin{aligned} &A_iA_{i+1}A_i - A_i = A_{i+1}A_iA_{i+1} - A_{i+1} \\ &A_iA_i = (t^{1/2} + t^{-1/2})A_i \\ &A_iA_j = A_jA_i \qquad \text{for } |i-j| \geq 2. \end{aligned} \tag{21}$$

Here we have introduced a parameter t through $t = \exp 2\eta$. We note that, for the case gl(2|0), $\{A_i\}$ satisfy the Temperley–Lieb algebra [29]. Using the operator A_i the Yang–Baxter operator $X_i(u)$ for the vertex model (5) is written as

$$X_i(u) = \rho(u)\,(I + f(u)A_i) \tag{22}$$

where

$$f(u) = \frac{\sinh u}{\sinh(\eta - u)} \tag{23}$$

$$\rho(u) = \frac{\sinh(\eta - u)}{\sinh\eta}. \tag{24}$$

If we define $G_i = I - t^{1/2}A_i$, then the operators $\{G_i\}$ generate the Hecke algebra [25]:

$$\begin{aligned} &G_iG_{i+1}G_i = G_{i+1}G_iG_{i+1} \\ &G_iG_i = (1-t)G_i + tI \\ &G_iG_j = G_jG_i \qquad \text{for } |i-j| \geq 2. \end{aligned} \tag{25}$$

We may regard the generators of the Hecke algebra as generalised Young operators [12, 44]. If we define

$$P_i^{[S]} = X_i(u = -\eta)/(t^{1/2} + t^{-1/2}) \tag{26}$$

$$P_i^{[A]} = X_i(u = \eta)/(t^{1/2} + t^{-1/2}) \tag{27}$$

then $P_i^{[S]}$ and $P_i^{[A]}$ are the projectors with even and odd parity, respectively. Hereafter we consider $P_i^{[S]}$. The projectors $P^{[k]}$ corresponding to the Young diagram with one row (k boxes) are recursively given by

$$P_i^{[k]} = P_i^{[k-1]}X_{i+k-2}(-(k-1)\eta)P_i^{[k-1]}. \tag{28}$$

Note that k is the number of strings in a composite string. The identity $I^{(k)}$ $(\in B_n^{[k]})$ is given by

$$I^{[k]} = P_1^{[k]} P_{k+1}^{[k]} \cdots P_{(n-1)k+1}^{[k]}. \tag{29}$$

Using the projectors we construct composite Yang–Baxter operators $\{Y_i^{[k]}(u); \text{for } i = 1, \cdots, n\}$ as [12, 14]

$$Y_i^{[k]}(u) = P_{(i-1)k+1}^{[k]} P_{ik+1}^{[k]} \left(\prod_{j=1}^{k} \hat{X}_i^{(j)}(u) \right) P_{(i-1)k+1}^{[k]} P_{ik+1}^{[k]} \tag{30}$$

where

$$\hat{X}_i^{(j)}(u) = \prod_{m=1}^{k} X_{ik+m-j}(u - (k - j - m + 1)\eta). \tag{31}$$

For example, in the case of two strings ($k = 2$) it is given by

$$\begin{aligned} Y_i^{[2]}(u) &= P_{2i-1}^{[2]} P_{2i+1}^{[2]} X_{2i}(u-\eta) X_{2i-1}(u) X_{2i+1}(u) X_{2i}(u+\eta) P_{2i-1}^{[2]} P_{2i+1}^{[2]} \\ &= X_{2i-1}(-\eta) X_{2i+1}(-\eta) X_{2i}(u-\eta) X_{2i-1}(u) X_{2i+1}(u) X_{2i}(u+\eta). \end{aligned} \tag{32}$$

The composite operators act on the composite space which is constructed by applying the identity operator $I^{[k]}$ to the space $V^{(1)} \otimes V^{(2)} \otimes \cdots \otimes V^{(kn)}$.

We note that this method of construction of composite solvable models is applicable not only for the vertex models related to sl(M) but also those associated with gl($M|N$), and furthermore any vertex or IRF models with quadratic minimal polynomials.

For the method of fusion there is an exception. Only for gl(1|1) does the composition yield degeneracy. The composite model of the free fermion model is equivalent to the free fermion model itself. We note that for the free fermion model the dimensions of the eigenspaces for the projection operators $P^{[S]}$ and $P^{[A]}$ are both equal to two. Therefore by the method of composition the dimensions of the edge variables do not increase.

3. Braid group and the Markov trace

3.1. Representations of the Braid group

We introduce braids and the braid group [45]. The braid group B_n is defined by a set of generators, $b_1, \cdots, b_{n-1}$ which satisfy

$$\begin{aligned} b_i b_{i+1} b_i &= b_{i+1} b_i b_{i+1} \\ b_i b_j &= b_j b_i \qquad |i-j| \geq 2. \end{aligned} \tag{33}$$

According to the general method [8, 9, 18–20] we construct braid matrices and the Markov trace on the representations. Taking the limit $u \to \infty$, we obtain representations of the braid group from the Boltzmann weights of the vertex models. The braid operator $G_i(+)$, the inverse operator $G_i(-)$ and the identity operator I are given by

$$G_i(\pm) = \lim_{u \to \infty} X_i(\pm u)/\rho(\pm u) \tag{34}$$

$$I = X_i(0). \tag{35}$$

Hereafter we write the matrix elements of the braid operator as

$$G^{ab}_{cd}(\pm) = \lim_{u\to\infty} w(c,d,b,a:\pm u)/\rho(\pm u). \tag{36}$$

Then we can express the braid operator (34) constructed from the Yang–Baxter operator as

$$G(\pm)_i = \sum_{abcd} G^{ab}_{cd}(\pm) I^{(1)} \otimes \cdots \otimes e^{(i)}_{ac} \otimes e^{(i+1)}_{bd} \otimes I^{(i+2)} \otimes \cdots \otimes I^{(n)}. \tag{37}$$

It may be instructive to write the matrix elements of the braid operator:

$$[G_i]^{a_1\cdots a_n}_{b_1\cdots b_n} = \prod_{j=1}^{i-1} \delta^{a_j}_{b_j} G^{a_i a_{i+1}}_{b_i b_{i+1}} \prod_{j=i+2}^{n} \delta^{a_j}_{b_j} \tag{38}$$

where δ^a_b is the Kronecker delta.

The elements of the braid matrices are given in the following:

$$\begin{aligned}
G^{aa}_{aa}(+) &= \begin{cases} 1 & \text{for } \epsilon_a = 1 \\ -t & \text{for } \epsilon_a = -1 \end{cases} \\
G^{ab}_{ab}(+) &= \begin{cases} 0 & \text{for } a < b \\ 1-t & \text{for } a > b \end{cases} \\
G^{ab}_{ba}(+) &= -t^{1/2} \qquad \text{for } a \neq b.
\end{aligned} \tag{39}$$

Here a variable t is defined by $t = \exp(2\eta)$. We obtain 2^{M+N} different representations depending on the choice of the signs $\{\epsilon_a\}$. [23]. Note that by replacing t with t^{-1} and multiplying the braid matrix by $-t$, we have an equivalent representation.

Each representation has only two eigenvalues 1 and $-t$. The braid matrices satify the Hecke algebra relations (25). Thus we have seen that the Hecke algebra also appears in the braid matrices associated with the Lie superalgebra gl$(M|N)$.

By taking the limit $\eta \to 0$ we derive the graded permutation operator (13) from the representation of the braid group (39). Thus the braid operator is a q analogue of the graded permutation operator.

Let us consider the connection of the braid matrix (39) to the Casimir operator of gl$(M|N)$. Pure braid generators $\{K_{ij}\}$ [45] are related to the generators of the braid group by

$$K_{ij} = b_i b_{i+1} \cdots b_{j-1} b_j^2 b_{j-1}^{-1} \cdots b_i^{-1} \qquad \text{for } i < j. \tag{40}$$

If the operator K_{ij} can be expanded in terms of a small parameter η as

$$K_{ij} = I + \eta\Omega_{ij} + \mathrm{O}(\eta) \tag{41}$$

then the operator Ω_{ij} is called the infinitesimal pure braid operator [30, 31, 46]. Note that the operator Ω_{ij} acts on the ith and jth vector spaces in the tensor product space $V^{(1)} \otimes \cdots \otimes V^{(n)}$. Let us introduce an operator C_{ij} by

$$C_{ij} = \sum_{ab} (-1)^{p(b)} E^{a\ (i)}_b E^{b\ (j)}_a \tag{42}$$

where $\{E_b^{a\,(i)}\}$ are the generators of gl$(M|N)$ acting on the ith vector space in the tensor product $V^{(1)}\otimes\cdots V^{(n)}$. The operator C_{ij} is equivalent to the Casimir operator of gl$(M|N)$ acting on the ith and jth vector spaces except for constant terms. The infinitesimal pure braid operator Ω_{ij} obtained from the braid matrix (39) coincides with the matrix elements of the operator C_{ij} except for a constant term, which is related to the normalisation factor of the braid matrix. Thus the the braid operator is related to the Casimir operator for gl$(M|N)$. In this sense the vertex model given by (5) is associated with gl$(M|N)$. Here we note that the matrix elements of the operator C_{ij} are calculated by using equation (12) for the graded tensor product:

$$(E\otimes F)(|a\rangle\otimes|b\rangle) = (-1)^{p(F)p(a)}E|a\rangle\otimes F|b\rangle. \tag{43}$$

It is remarked that through the symmetry breaking transformation (8) we can derive different braid matrices from the graded vertex model (5) [22]. The number of the non-zero elements are different.

3.2. General form of the Markov trace

It is known that any oriented link can be expressed by a closed braid. The equivalent braids expressing the same link are mutually transformed by a finite sequence of two types of operations, Markov moves I and II (figure 2). The Markov trace $\phi(\cdot)$ is a linear functional on the representation of the braid group that has the following properties (the Markov properties):

$$\text{I}\qquad \phi(AB)=\phi(BA)\qquad A,B\in B_n \tag{44}$$

$$\text{II}\qquad \begin{aligned}&\phi(Ab_n)=\tau\phi(A)\\ &\phi(Ab_n^{-1})=\bar\tau\phi(A)\qquad A\in B_n, b_n\in B_{n+1}\end{aligned} \tag{45}$$

where

$$\begin{aligned}\tau&=\phi(b_i)\qquad \text{for all } i\\ \bar\tau&=\phi(b_i^{-1})\qquad \text{for all } i.\end{aligned} \tag{46}$$

From the Markov trace we obtain a link polynomial $\alpha(\cdot)$ as [8–10, 12–14, 18–20]

$$\alpha(A) = (\tau\bar\tau)^{-(n-1)/2}(\bar\tau/\tau)^{(1/2)e(A)}\phi(A)\qquad A\in B_n. \tag{47}$$

Here $e(A)$ is the exponent sum of b_i in the braid A, which is equivalent to the writhe of the link diagram. For instance, if $A=b_1^4b_2^{-2}b_3b_1^{-1}$, then $e(A)=4-2+1-1=2$.

Let us construct the Markov trace on the representations derived in the last section. We find that for any grading $\{\epsilon_i\}$ the Markov trace is given by

$$\phi(A)=\frac{\mathrm{Tr}\,(H(n)A)}{\mathrm{Tr}\,(H(n))}\qquad A\in B_n\qquad [H(n)]_{b_1b_2\cdots b_n}^{a_1a_2\cdots a_n}=\prod_{j=1}^{n}h(a_j)\delta_{b_j}^{a_j}. \tag{48}$$

Here the diagonal matrix h is

$$h(j)=\epsilon_j\exp\left(\eta(\sum_{k=1}^{j-1}2\epsilon_k+\epsilon_j-M+N)\right)\qquad \text{for } j=1\cdots M+N. \tag{49}$$

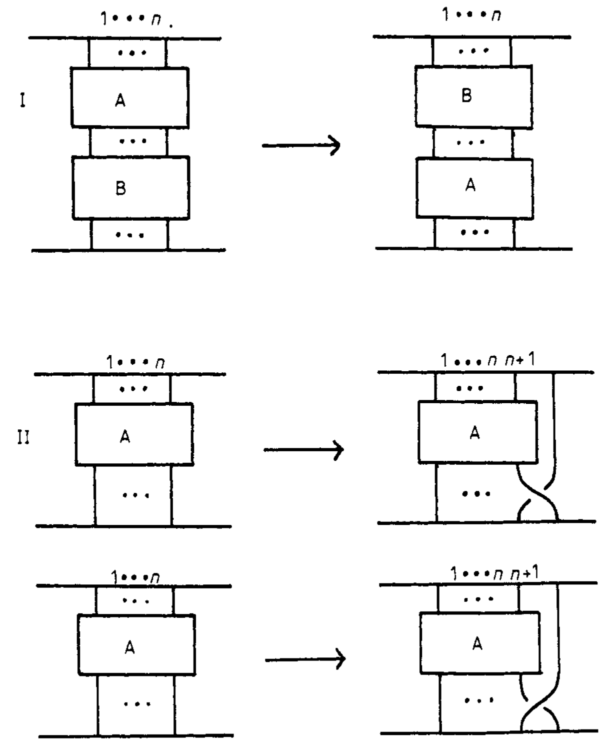

Figure 2. Markov moves I and II.

We give sufficient conditions for the Markov properties (44) and (45) in the following. For the Markov property I the charge conservation property is sufficient. For the Markov property II the charge conservation property and the following condition are sufficient:

$$\sum_b G^{ab}_{ab}(\pm)h(b) = \chi(\pm) \qquad \text{(independent of } a). \tag{50}$$

Here the τ factors are related to $\chi(\pm)$ as $\bar{\tau}/\tau = \chi(-)/\chi(+)$.

We can prove the matrix h given by (49) satisfies condition (50) by induction on $M+N$. Thus we have constructed the Markov trace on the braid matrix given in (39). We remark that in the limit $\eta \to 0$, the Markov trace reduces to the supertrace [39] $\operatorname{str} A = \sum_j \epsilon_j A_{jj}$. Hence the Markov trace (48) is an extension (q analogue) of the supertrace.

From the explicit form of the Markov trace we find that

$$\chi(\pm) = \exp\{\pm(M-N-1)\eta\}. \tag{51}$$

If we define $q^{1/2} = \sum_j h(j)$, then

$$q^{1/2} = \frac{\sinh((M-N)\eta)}{\sinh \eta}. \tag{52}$$

We can prove the extended Markov property [13, 15, 18–20], which is an extension of the Markov property with finite spectral parameter:

$$\sum_b X^{ab}_{ab}(u)h(b) = H(u;\eta)\rho(u) \qquad \text{(independent of } a) \tag{53}$$

where the function $H(u;\eta)$ is called the characteristic function and given by

$$H(u;\eta) = \frac{\sinh((M-N)\eta - u)}{\sinh(\eta - u)}. \tag{54}$$

This form is a generalisation of the characteristic function for the model of the type A_{M-1} (sl(M)) [15, 18–20].

We can generalise the formula (48) for the Markov trace into the following:

$$\phi^*(A) = \frac{\mathrm{Tr}\,(H^*(n)A)}{\mathrm{Tr}\,(H^*(n))} \qquad A \in B_n \qquad [H^*(n)]^{a_1 a_2 \cdots a_n}_{b_1 b_2 \cdots b_n} = k(a_1)\delta^{a_1}_{b_1} \prod_{j=2}^{n} h(a_j)\delta^{a_j}_{b_j}. \tag{55}$$

Here $k(a)$ is arbitrary. Note that the Markov move II always operates on the braids from the right and therefore the ends of the left-most string keeps untouched in the operation. In this way we have the Markov trace $\phi^*(\cdot)$. This regularisation is useful when the quantity $q^{1/2} = \sum_j h(j)$ is equal to 0. In fact $q^{1/2}$ is equal to 0 for gl($M|M$). In this case using this regularisation (55) we define the Markov trace as

$$\phi^*(A) = \mathrm{Tr}\,(H^*(n)A)/\Sigma_j k(j) \qquad A \in B_n. \tag{56}$$

For the representation of the braid group (39) the value of the Markov trace $\phi^*(\cdot)$ (55) is independent of the choice of $\{k(j)\}$ except for special cases such as $\sum_j k(j) = 0$.

3.3. Composite string representation of the braid group

Let us consider the braid operators derived from the composite solvable models. In the limit $u \to \infty$ we have

$$G_i^{[k]} = Y_i^{[k]}(u=\infty) = P^{[k]}_{(i-1)k+1} P^{[k]}_{ik+1} \left(\prod_{j=1}^{k} \hat{G}_i^{(j)} \right) P^{[k]}_{(i-1)k+1} P^{[k]}_{ik+1} \tag{57}$$

where

$$\hat{G}_i^{(j)} = \prod_{m=1}^{k} G_{ik+m-j}. \tag{58}$$

The eigenvalues of the composite braid operator determine the reduction relation (minimal polynomial) of the operator [8–10, 12, 18–20]. Since the calculation done in [12] was based only on the defining relations of the Hecke algebra, we readily see that the eigenvalues $\{c_r\}$ of the braid operators derived from the composite Yang–Baxter operators are given by [12]

$$c_r = (-1)^{r+2s+1} t^{s(s+1)-r(r+1)/2} \qquad r = 0, 1, \cdots k. \tag{59}$$

Here the number s is given by $2s = k$. We have chosen the normalisation factor so that the eigenvalues coincide with those obtained in [8–12].

3.4. Link polynomials

The link polynomial obtained from the vertex model associated with $\mathrm{gl}(M|N)$ has the skein relation

$$\alpha(L_+) = t^{l/2}(1-t)\alpha(L_0) + t^{l+1}\alpha(L_-). \tag{60}$$

Here we have defined a number l as

$$l = M - N - 1. \tag{61}$$

Since the skein relation is of second degree, the link polynomial is fully determined by the relation. Thus we obtain a hierarchy of link polynomials that depends on the number $l = M - N - 1$. The most characteristic point of this hierarchy is that from the Markov traces and the braid matrices with different sizes the same link polynomial for an integer l is constructed [23]. From different models related to $\mathrm{gl}(M|N)$ with $l = M - N - 1$ we obtain the same link polynomial. Note that the hierarchy includes the case $l = 0$ where $\bar{\tau}/\tau = 1$. For any integer l we have a link polynomial with the relation (60). We remark that the link polynomial for an integer l corresponds to that for $-2 - l$ under the replacement of t by $1/t$.

The HOMFLY polynomial [26, 27] is characterised by the second-degree skein relation:

$$\alpha(L_+) = \omega^{1/2}(1-t)\alpha(L_0) + \omega t\alpha(L_-). \tag{62}$$

Here t and ω are independent (continuous) variables. We see that the link polynomials constructed from the $\mathrm{gl}(M|N)$ type vertex models correspond to the cases $\omega = t^l$, $l \in \mathbf{Z}$ of the HOMFLY polynomial. Based on the Markov traces we thus obtain a hierarchy of link polynomials corresponding to the HOMFLY polynomial [23].

We discuss special cases in the hierarchy. The link polynomial for $l = -1$ is the Alexander polynomial [24]. The case $l = 1$ corresponds to the Jones polynomial. [25]. Therefore we have a number of braid matrices with different sizes which lead to the Alexander polynomial and the Jones polynomial. Braid group representations and link polynomials for the cases $N = 0$ were obtained from the study of the $\mathrm{sl}(M)$ vertex models [47], from q deformation of universal enveloping algebras $U_q(\mathrm{sl}(M))$ [48], and by using the Wu–Kadanoff–Wegner transformation [13, 19, 20] from the A_{M-1} type IRF models [15]. The braid matrix for the case $M = 1, N = 1$ was also obtained by solving the definfing relations of the braid group [49]. This case is obtained from the free fermion 6-vertex model in our approach.

Let us consider link polynomials derived from the composite models. We can show the existence of the Markov trace for the composite string representations by two different methods. The first one is to prove the extended Markov property for the composite model. From the operator form of the composite Yang–Baxter operator $Y_i^{[k]}(u)$ (30) (for k strings) the characteristic function $H^{[k]}(u;\eta)$ is recursively calculated as

$$H^{[k]}(u;\eta) = \prod_{r=1}^{k} \frac{\sinh((M-N-1+r)\eta - u)}{\sinh(r\eta - u)}. \tag{63}$$

The other method is to consider the composite string representation. [11, 12, 20]. By the discussion given in [12] a sufficient condition for the Markov property is the following eigenvalue equation:

$$P_i^{[\lambda]}\Delta_i^2 = \alpha_\lambda P_i^{[\lambda]} \tag{64}$$

where Δ_i is half twist [12, 45] and λ denotes the symmetry of the projector. We construct link polynomials by solving equation (64). Projectors satisfying equation (64) for mixed symmetry have been calculated [12, 19, 20]. The braid operators for the composite string are given by

$$G_i^{[\lambda]} = P_{(i-1)k+1}^{[\lambda]} P_{ik+1}^{[\lambda]} \left(\prod_{j=1}^{k} \hat{G}_i^{(j)} \right) P_{(i-1)k+1}^{[\lambda]} P_{ik+1}^{[\lambda]} \tag{65}$$

where

$$\hat{G}_i^{(j)} = \prod_{m=1}^{k} G_{ik+m-j}. \tag{66}$$

The Markov trace $\psi^{[\lambda]}(\cdot)$ is given by (for the cases $\mathrm{gl}(M|N)$ with $M \neq N$) [11, 12]

$$\psi^{[\lambda]}(A) = \phi(A)/[\phi(P_i^{[\lambda]})]^n \qquad A \in B_n^{[\lambda]}. \tag{67}$$

Here $\phi(\cdot)$ is defined by (48) and (49). For the cases $\mathrm{gl}(M|M)$, we introduce the Markov trace by

$$\psi^{[\lambda]}(A) = \phi^*(A) \qquad A \in B_n^{[\lambda]} \tag{68}$$

where $\phi^*(\cdot)$ is given by (56). We remark that we can also use the definition (68) for the cases $\mathrm{gl}(M|N)$ with $M \neq N$.

Link polynomials are given by

$$\alpha^{[\lambda]}(L) = (Z_\lambda \bar{Z}_\lambda)^{-(n-1)/2} \left(\frac{\bar{Z}_\lambda}{Z_\lambda} \right)^{e(A)/2} \psi^{[\lambda]}(A) \qquad A \in B_n^{[\lambda]} \tag{69}$$

where A is a braid whose closed braid is equivalent to the link L, $e(A)$ is the exponent sum of the braid A and

$$Z_\lambda = \psi^{[\lambda]}(G_j) \qquad G_j \in B_n^{[\lambda]} \tag{70}$$

$$\bar{Z}_\lambda = \psi^{[\lambda]}(G_j^{-1}) \qquad G_j^{-1} \in B_n^{[\lambda]}. \tag{71}$$

The skein relations for the link polynomials constructed from the composite models with symmetry corresponding to the Young diagram for one row is given as follows:

$$\alpha(L_{3+}) = t^l(1 - t^2 + t^3)\alpha(L_{2+}) + t^{2l}(t^2 - t^3 + t^5)\alpha(L_+) - t^{3l+5}\alpha(L_0) \qquad \text{for } k = 2 \tag{72}$$

$$\begin{aligned}\alpha(L_4+) = t^{3l/2}(1 - t^3 + t^5 - t^6)\alpha(L_{3+}) + t^{3l}(t^3 - t^5 + t^6 + t^8 - t^9 + t^11)\alpha(L_{2+})\\ + t^{9l/2}(-t^8 + t^9 - t^11 + t^14)\alpha(L_+) - t^{6l}t^{14}\alpha(L_0) \qquad \text{for } k = 3.\end{aligned} \tag{73}$$

We see that these link polynomials correspond to the two-variable link invariants with $\omega = t^l$ [12, 20].

It is remarked that the factor $\omega = t^l$ is different from the $\mathrm{sl}(M)$ cases for $l = 0, -1$ and therefore the link polynomials for $l = 0, -1$ are new in this sense. It is interesting that the Alexander polynomial is related to the free fermion model. Note that generalisations of the Alexander polynomial are obtained from the composite models related to $\mathrm{gl}(M|M)$ ($M > 1$). These link polynomials have higher-degree skein relations such as (72) and (73).

4. Concluding remarks

A family of vertex models given in (5) may be regarded as models associated with $gl(M|N)$ for the following reasons: (i) the braid matrix derived from the model is a q analogue of the graded permutation operator; (ii) the infinitesimal pure braid operator for the braid matrix is equivalent to the Casimir operator of $gl(M|N)$; (iii) the rational solution for the vertex model obtained in the limit $\eta \to 0$ is written in terms of the graded permutation operator.

Based on the Markov trace we have constructed link polynomials from the vertex models associated with $gl(M|N)$ and the composite models. In particular, we have a hierarchy of link polynomials with $\omega = \bar{\tau}/\tau = 1, -t$, which are not contained in those related to $sl(M)$. Using our knowledge of the Hecke algebra we can compare the link polynomials with the two-variable link invariants [11, 12]. The link polynomials are one-variable restrictions $\omega = t^l$ of the two-variable link invariants.

It should be emphasised that the braid matrix (39) derived from the vertex model associated with $gl(M|N)$ satisfies the Hecke algebra. Under the correspondence (17) the braid matrix is equivalent to the R matrix [50] discussed in the context of the quantum group, or q analogue of universal enveloping algebra. The most important information for the construction of link poylnomials is the fact that the braid operators satisfy the Hecke algebra.

It is interesting to note that there are different braid matrices with the same eigenvalues and number of states, such as the matrix related to $gl(2|0)$ (or $sl(2)$) and that for $gl(1|1)$. Let us consider things from the viewpoint of solving the defining relation of the braid group (the Yang–Baxter relation). We can show that the solution of the defining relation of the braid group is unique except for signs ($G_{ba}^{ab} = \pm t^{1/2}$, for $a \neq b$) and the grading under the following assumptions. (i) The braid matrix has the charge conservation property. (ii) If $\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{c} + \boldsymbol{d}$, then $\boldsymbol{a} = \boldsymbol{c}$ and $\boldsymbol{b} = \boldsymbol{d}$, or $\boldsymbol{a} = \boldsymbol{d}$ and $\boldsymbol{b} = \boldsymbol{c}$. (iii) The eigenvalues are 1 and $-t$. (iv) The braid matrix is symmetric. (v) $G_{ab}^{ab} = 0$, for $a < b$.

We may consider the braid matrix from the viewpoint of the monodromy of the Knizhnik–Zamolodchikov equation. The Knizhnik–Zamolodchikov equation associated with the Lie superalgebras are given by

$$\kappa \frac{\partial \Phi}{\partial z_i} = \sum_{j \neq i} \frac{C_{ij}}{z_j - z_i} \Phi. \tag{74}$$

The operator C_{ij} is

$$C_{ij} = \boldsymbol{t}_i \cdot \boldsymbol{t}_j = \sum_a (-1)^{p(a)} t_a^{(i)} t_a^{(j)} \tag{75}$$

where $\{t_a^{(i)}\}$ are generators of the algebra acting on the ith vector space in the tensor product $V^{(1)} \otimes \cdots V^{(n)}$. Note that for $gl(M|N)$ the matrix elements of the operator C_{ij} coincides with the infinitesimal pure braid operator Ω_{ij} derived from the braid operators. The infinitesimal pure braid relations for the infinitesimal pure braid operator Ω_{ij} give the integrability condition of the Knizhnik–Zamolodchikov equation. [30, 31, 46]. Therefore we have shown the integrability of the equation from the viewpoint of the solvable models. The knowledge of the braid matrix also determines the monodromy matrix related to $gl(M|N)$.

We can consider solvable models related to other Lie superalgebras. The problem is to obtain explicit forms of the braid matrices and construct associated solvable models [51–53].

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